(Geometric) representation theory: exercises to an introduction

Travis Schedler

February 2, 2025

1. Prove directly that every finite-dimensional real representation of S^1, or finite-dimensional holomorphic representation of $\mathbb{C}^\times$, is isomorphic to a direct sum of one-dimensional representations of the form $z \mapsto z^m$ for some $m \in \mathbb{Z}$.

2. Prove directly that every finite-dimensional real representation of the Lie algebra $\mathbb{R}$, or every finite-dimensional complex representation of $\mathbb{C}$, is isomorphic to $\lambda \mapsto \lambda A$ for some square matrix A, and two of these are isomorphic if and only if the matrices are conjugate. Using Lie's theorem, deduce that every finite-dimensional real or complex representations of the Lie groups $\mathbb{R}$ or $\mathbb{C}$, respectively, are given by $\lambda \mapsto e^{\lambda A}$ for a square matrix A, with $e^{\lambda A}, e^{\lambda A'}$ isomorphic if and only if A is conjugate to A'.

3. Classify directly the irreducible representations of the complex Lie algebra $\mathfrak{sl}_2(\mathbb{C})$, following the example in the notes, and use this to classify holomorphic representations of $\mathsf{SL}_2(\mathbb{C})$.

4. Show that this classification matches the one given by the Borel–Weil theorem.

5. The notes mention the deep fact that if G is a semisimple complex Lie group and V is a finite-dimensional representation, then the image of $\rho_V : G \to \mathsf{GL}(V)$ is closed. Show from this that if G is noncompact and almost simple (meaning $Z(G)$ is finite and $G/Z(G)$ is simple, i.e., has no proper nontrivial closed normal subgroup), and V is nontrivial, then there can be no G-invariant unitary structure on V.

6. Suppose that $\pi : X \to Y$ is a $\mathbb{P}^1$-bundle. Using derived functors, show that for a line bundle L on Y, we have $H^i(Y, L) \cong H^i(X, \pi^* L)$. Apply this in the case $G = \mathsf{SL}_3$, to
$$B = \left\{ \begin{pmatrix} a & * & * \\ 0 & b & * \\ 0 & 0 & (ab)^{-1} \end{pmatrix} \right\} \subseteq P = \left\{ A = \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \middle| \det A = 1 \right\}.$$
Show that $G/P \cong \mathbb{P}^2$ and that $G/B \to G/P$ is a $\mathbb{P}^1$-bundle, and deduce from the Borel–Weil–Bott theorem the cohomology of all line bundles on $\mathbb{P}^2$. Bonus: extend this to the irreducible G-equivariant vector bundles on $\mathbb{P}^2$. Are these all the irreducible vector bundles on $\mathbb{P}^2$?

7. In this problem you will construct the proof that G-equivariant and ordinary line bundles on $X = G/B$ are equivalent. This uses the reformulation of the G action on L as an isomorphism $a^*L \cong \pi_2^*L$ for $a : G \times X \to X$ the action and $\pi_2 : G \times X \to X$ the second projection, which satisfies the so-called cocycle condition (i.e. that it actually defines an action).

   This uses the following properties of G and X:

   (a) X is projective (proper would be enough), so that global functions are constant, and hence a line bundle on $G \times X$ which is trivial on $\{g\} \times X$ for all $g \in G$ is isomorphic to π_1^*G for $\pi_1 : G \times X \to G$ the projection;

   (b) X is simply-connected (thanks to a cell decomposition for any semisimple G), thus $H^1(X, \mathbb{C}) = 0$, and X is projective, so by the Hodge decomposition, $H^1(X, \mathcal{O}) \subseteq H^1(X, \mathbb{C})$ is also zero, hence $\mathsf{Pic}(X)$ is discrete (the tangent space is $H^1(X, \mathcal{O})$ in general);

   (c) G is affine, hence $H^1(G, \mathcal{O}) = 0$, so $\mathsf{Pic}\, G$ is also discrete; moreover, $H^2(G, \mathbb{Z}) = 0$, so that $\mathsf{Pic}\, G$ is actually zero;

   (d) G is perfect, i.e., $[G, G] = G$, which follows because G is connected semisimple.

   If you are stuck you could look at Lurie's proof of the Borel–Weil–Bott theorem.